Problem 1: [Some Basic Properties (4P+4P) ]

Prove or disprove each of the following claims:

- For arbitrary ideals $I, J \subseteq \mathbb{Q}[x_1, \ldots, x_n]$, we have $\sqrt{I} \cdot \sqrt{J} = \sqrt{I \cdot J}$.
- For arbitrary $f_1, \ldots, f_r \in \mathbb{Q}[x_1] \setminus \{0\}$ (r $\in \mathbb{N}$), the set
  \[ G := \{\gcd(f_1, \ldots, f_r)\} \]
  is a Gröbner basis of the ideal $\langle f_1, \ldots, f_r \rangle \subseteq \mathbb{Q}[x_1]$.

Problem 2: [Subalgebras of a polynomial ring (6P+6P) ]

Consider the polynomials $f_1 := x_1 + x_2 + x_3$, $f_2 := x_1^2 + x_2^2 + x_3^2$ and $f_3 := x_1 \cdot x_2 \cdot x_3$ in $\mathbb{Q}[x_1, x_2, x_3]$.

- Prove that $\mathbb{Q}[f_1, f_2, f_3]$ and $\mathbb{Q}[x_1, x_2, x_3]$ are isomorphic (as $\mathbb{Q}$-algebras).
- Show that $f_4 := x_1^4 + x_2^4 + x_3^4 \in \mathbb{Q}[f_1, f_2, f_3]$ by explicitly giving a representation of $f_4$ as a polynomial in $f_1, f_2, f_3$.

Problem 3: [Ideals and Varieties (6 P) ]

Prove that the curve $V(z - x^3, y - x^2) \subseteq \mathbb{R}^3$ is contained in the surface $V(x \cdot z - y^2) \subseteq \mathbb{R}^3$.

Problem 4: [Meeting Combinatorics (9 P) ]

Consider a finite undirected graph $G := (V, E)$ without loops. A three coloring of the vertices of $G$ is a map $c : V \rightarrow \{1, 2, 3\}$ such that $c(v_1) \neq c(v_2)$ for all $\{v_1, v_2\} \in E$.

Construct an ideal $I_G$ such $V(I_G) \neq \emptyset$ if and only if $G$ has a three coloring.

Hint: You can use a polynomial ring over $\mathbb{F}_2$ with 3-$|V|$ indeterminates.

Good luck & have fun!!!