MATH DAY 2009 at FAU

Competition B-Teams

SOLUTIONS

1. What are the two rightmost digits of the number 2009^{2009}?

(A) 09 **(B)** 49 **(C)** 61 **(D)** 89 **(E)** NA

SOLUTION. We work mod 100; $2009 \equiv 9$ mod 100, therefore $2009^{2009} \equiv 9^{2009}$ mod 100. If we start computing powers and reducing mod 100, we get

$$9^2 \equiv 81,\ 9^3 = 729 \equiv 29,\ 9^4 \equiv 29 \times 9 = 261 \equiv 61,\ 9^5 \equiv 61 \times 9 = 549 \equiv 49,\ 9^6 \equiv 49 \times 9 = 441 \equiv 41,$$
$$9^7 \equiv 41 \times 9 = 369 \equiv 69,\ 9^8 \equiv 69 \times 9 = 621 \equiv 21,\ 9^9 \equiv 21 \times 9 = 189 \equiv 89,\ 9^{10} \equiv 89 \times 9 = 801 \equiv 1$$

and we can stop! We have all the information we need. We see

$$2009^{2009} \equiv 9^{2009} = \left(9^{10}\right)^{200} 9^9 \equiv 1^{200} 89 = 89.$$

The correct solution is **D.**

Note: Knowing Euler's Theorem, that $a^{\phi(n)} \equiv 1$ mod n if a, n have no common factor other than 1, can shorten the number of computations. In fact, $\phi(100) = 40$ so that

$$2009^{2000} = \left(2009^{40}\right)^{50} \equiv 1^{50} = 1, \text{ hence } 2009^{2009} \equiv 2009^9 \equiv 9^9,$$

and to compute 9^9 mod 100 we can go in two steps; first 9^3 reduced mod 100, then get 9^9 as the cube of 9^3. ■

2. The largest positive integer n such that $n + 5$ divides $n^3 + 5$

(A) is a multiple of 10 **(B)** satisfies $25 < n < 75$ **(C)** satisfies $75 < n < 120$
(D) satisfies $120 < n < 375$ **(E)** satisfies $375 < n < 7505$

SOLUTION. By long division, or otherwise, $n^3 + 5 = (n+5)(n^2 - 5n = 25) - 120$ and it follows $n + 5$ divides $n^3 + 5$ if and only if it divides 120. The largest integer dividing 120 is 120 itself, thus $n + 5 = 120$, $n = 115$ is the largest integer such that $n + 5$ divides $n^3 + 5$.

The correct solution is **C.** ■

3. How many integers n such that $100 < n < 10{,}00$ are divisible by 2 but not by 6?

Write your answer directly onto the answer sheet.

SOLUTION. n ranges from 101 to 9999. Of these, $(9998 - 100)/2 = 4949$ are even. The multiples of 6 in the given range are of the form $6k$ with $100/6 < k < 10000/k$ from which (k being an integer) $17 \le k \le 1666$; a total of 1650 possible values for k. Since all the multiples of 6 are even, it follows that the number of even numbers not divisible by 6 in the given range is $4949 - 1650 = 3299$.

The number to be entered should be **3299.** ■

4. Suppose that $r_1, r_2, \ldots, r_{100}$ are the one hundred (possibly complex) roots of the equation $x^{100} - 4x + 5 = 0$. (You may assume all the roots are distinct). Compute the sum

$$r_1^{100} + r_2^{100} + \cdot + r_{100}^{100}$$

of the 100th powers of all the roots.

(A) -99 **(B)** 500 **(C)** -599 **(D)** 599 **(E)** -500

SOLUTION. If r is a root of the equation, then $r^{100} = 4r - 5$. Thus the sum of all the 100th powers of the roots equals 4 times the sum of the roots minus 100×5. The sum of all the roots of a polynomial of degree n equals (-1) times the coefficient of the term of degree $n-1$. In this case, with $n = 100$, it would be the term of degree 99, which is 0. We conclude that the sum of all the 100th powers is -500.

The correct solution is **E.** $\blacksquare$

5. If x is a real number, then $[x]$ denotes the largest integer less than or equal to x. For example, $[5] = 5$, $[\sqrt{2}] = 1$, $[-3] = -3$, $[-2.5] = -3$. Which of the following gives an **integer** that is **always** closest to x:

(A) $-\left[-x + \frac{1}{2}\right]$ **(B)** $2\left[\frac{x+1}{2}\right]$ **(C)** $[x] + \frac{1}{2}$ **(D)** $-[-x]$ **(E)** NA

SOLUTION. We can discard **C** at once; it is not an integer. Suppose $x = m + b$, where m is an integer and $0 \le b < 1$. We see that

$$-\left[-x+\frac{1}{2}\right] = -\left[-m-b+\frac{1}{2}\right] = \begin{cases} -(-m-1) = m+1 & \text{if } b > 1/2, \\ -(-m) = m & \text{if } b \le 1/2. \end{cases}$$

$$2\left[\frac{x+1}{2}\right] = \left[\frac{m+b+1}{2}\right] = \begin{cases} m & \text{if } m \text{ is even,} \\ m+1 & \text{if } m \text{ is odd.} \end{cases}$$

$$-[-x] = -[-m-b] = \begin{cases} m & \text{if } b = 0, \\ m+1 & \text{otherwise.} \end{cases}$$

Clearly, the first choice is always the best.

The correct solution is **A.** $\blacksquare$

6. Suppose that $0 < b < a$ and the sum $a+b$, the product ab, and the difference of squares $a^2 - b^2$ are all equal. Determine a.

(A) 2 **(B)** $\sqrt{5}$ **(C)** $\frac{\sqrt{5}+1}{2}$ **(D)** $\frac{\sqrt{5}+3}{2}$ **(E)** NA

SOLUTION. From $a + b = a^2 - b^2 = (a+b)(a-b)$ we conclude that $a - b = 1$ or $b = a - 1$. Using this in $a + b = ab$ we get $2a - 1 = a^2 - a$, hence $a^2 - 3a + 1 = 0$. The roots of this quadratic equation are $a = (3 \pm \sqrt{5})/2$. While both are positive, only the larger one also results in $b = a - 1 > 0$. Thus $a = (3 + \sqrt{5})/2$, $b = (1 + \sqrt{5})/2$. As a final check, one can see that, with this choice, $a + b, ab, a^2 - b^2$ all equal $2 + \sqrt{5}$.

The correct solution is **D.** $\blacksquare$

7. The equation

$$x^3 + ax^2 + bx + c = 0$$

has the roots $x = 1$, $x = 2$, and $x = 3$. Determine b.

Write your answer directly onto the answer sheet.

SOLUTION. We'll have $x^3 + ax^2 + bx + c = (x-1)(x-2)(x-3)$, from which it follows that $b = 1\cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11$.

The number to be entered should be **11.** $\blacksquare$

8. Compute the sum

$$\binom{2009}{1}\sin 60^\circ + \binom{2009}{2}\sin 120^\circ + \binom{2009}{3}\sin 180^\circ + \cdots + \binom{2009}{2009}\sin(2009 \times 60^\circ).$$

(A) $-(\sqrt{3})^{2009}$ **(B)** $-\frac{1}{2}(\sqrt{3})^{2009}$ **(C)** $\frac{1}{2}(\sqrt{3})^{2009}$ **(D)** $(\sqrt{3})^{2009}$ **(E)** 0 **(F)** NA

Note: If m, n are non-negative integers, $m \le n$, then

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{m!}.$$

In this context one interprets $0! = 1$ so that $\binom{n}{0} = \binom{n}{n} = 1$. One can also define $\binom{n}{m}$ as being the m-th entry of the n-th row of the Pascal triangle, the count beginning with 0.

SOLUTION. We will use Euler's formula $e^{ix} = \cos x + i \sin x$, $\sin x = \Im(e^{ix})$ if x is real, where $\Im(z)$ denotes the imaginary part of the complex number z. This formula requires x to be in radians, so we'll need to substitute $\pi/3$ for 60. We also need to recall Newton's binomial formula,

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n} b^n.$$

Now, (notice that the added first term in the second line has imaginary part 0)

$$\begin{aligned} S &= \binom{2009}{1}\sin 60^\circ + \binom{2009}{2}\sin 120^\circ + \binom{2009}{3}\sin 180^\circ + \cdots + \binom{2009}{2009}\sin(2009 \times 60^\circ) \\ &= \Im\left(\binom{2009}{0} e^{0\pi i/3} + \binom{2009}{1} e^{\pi i/3} + \binom{2009}{2} e^{2\pi i/3} + \cdots + \binom{2009}{2009} e^{2009\pi i/3}\right) \\ &= \Im\left((1+e^{\pi i/6})^{2009}\right) = \Im\left((1+\cos 60^\circ + i\sin 60^\circ)^{2009}\right). \end{aligned}$$

To solve this, we need to recall that $1 + \cos x = 2\cos^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$ so that

$$1 + \cos 60^\circ + i \sin 60^\circ = 2\cos 30^\circ(\cos 30^\circ + i \sin 30^\circ)$$

and by DeMoivre's formula

$$(1+\cos 60^\circ + i\sin 60^\circ)^{2009} = 2^{2009}(\cos 30^\circ)^{2009}(\cos(2009 \times 30)^\circ + i\sin(2009 \times 30)^\circ)$$

It is not too hard to see that $\cos(2009 \times 30)^\circ = -\cos 30^\circ = -\sqrt{3}/2$, $\sin(2009 \times 30)^\circ = \sin 30^\circ = 1/2$ so that

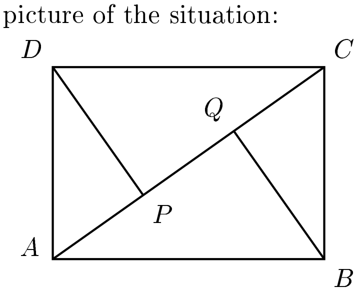
$$S = \Im\left(2^{2009}\left(\frac{\sqrt{3}}{2}\right)^{2009}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\right) = \frac{1}{2}(\sqrt{3})^{2009}.$$

The correct solution is **C.** ■

9. $ABCD$ is a rectangle in which the shorter side AD has length 1. The perpendiculars from B and D to the diagonal AC divide the diagonal into three equal parts. Find the length of AB.

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** $\sqrt{5}$ **(D)** 3 **(E)** NA

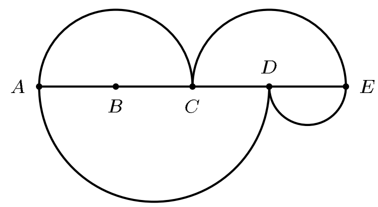
SOLUTION. We have the following picture of the situation:



Let $x = |AB|$ and let $d = |AP| = |AC|/3$. By Pythagoras, $|DP|^2 = 1 - d^2$; also by Pythagoras, $x^2 = |CD|^2 = |DP|^2 + |PC|^2 = 1 - d^2 + 4d^2 = 1 + 3d^2$, so $d^2 = (x^2 - 1)/3$. On the other hand, again by Pythagoras, $x^2 + 1 = |AB|^2 + |BC|^2 = |AC|^2 = 9d^2$. Thus $x^2 + 1 = 3(x^2 - 1)$, hence $x^2 = 4$ or $x = 2$.

The correct solution is **A.** $\blacksquare$

10. A line AE is divided into four equal parts by the points B, C, D. Semicircles are drawn with segments AC CE, AD and DE as diameters.



The ratio of the area enclosed above the line AE to the area enclosed below the line is

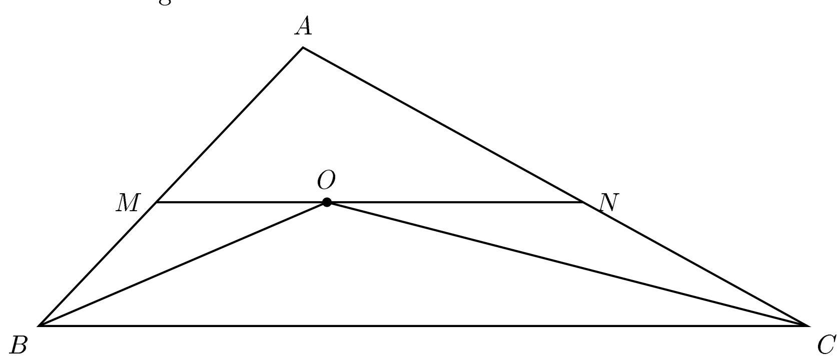
(A) $4:5$ **(B)** $5:4$ **(C)** $1:1$ **(D)** $8:9$ **(E)** NA

SOLUTION. The areas of the circles are proportional to the squares of their diameters. Considering as one unit one fourth of the line AE, the sum of the areas above the line is proportional to 8, the one below to 10.

Thus: Area above : Area below = 8:10 = 4 : 5.

The correct solution is **A.** $\blacksquare$

11. In the picture below, BO bisects the angle $\angle ABC$ and CO bisects $\angle ACB$. If $|AB| = 4$, $|BC| = 8$, and $AC = 6$, determine the perimeter of the triangle AMN.



(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** NA

SOLUTION. Notice that the triangle $\triangle BOM$ is isosceles. In fact, let $\alpha = \angle OBC = \angle OBM$, then

$$\begin{aligned} \angle AMN &= \angle ABC = 2\alpha, \\ \angle BMN &= 180^\circ - \angle AMN = 180^\circ - 2\alpha, \\ \angle MOB = 180^\circ - (\alpha + 180^\circ - 2\alpha) = \alpha. \end{aligned}$$

The assertion follows. We conclude that $|MO| = |MB|$. Similarly, $|ON| = |NC|$. Thus

Perimeter of $\triangle AMN$

$$= |AM| + |MN| + |NA| = |AM| + |MO| + |ON| + |NA| = |AM| + |MB| + |NC| + |NA| = |AB| + |AC| = 10.$$

The correct solution is **C.** $\blacksquare$

12. Which of the following conditions does **NOT** guarantee that the convex quadrilateral $ABCD$ is a parallelogram?

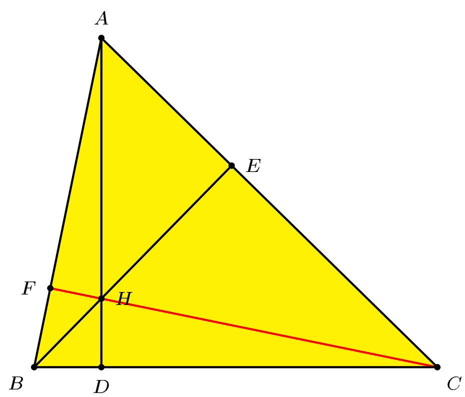
(A) $AB = CD$ and $AD = BC$ **(B)** $\angle A = \angle C$ and $\angle B = \angle D$ **(C)** $AB = CD$ and $\angle A = \angle C$
(D) $AB = CD$ and $AB//CD$ **(E)** NA

SOLUTION. Start with a triangle ABD with small angle A. Construct the arc BAD and reflect about BD. All points C on the reflected arc will have the property that in the quadrilateral $ABCD$, $\angle C = \angle A$. There are usually **two** points C on this arc with $CD = AB$.

The correct solution is **C.** ■

13. In triangle ABC, the altitude from A to BC meets BC at D, and the altitude from B to CA meets AD at H. If $AD = 4$, $BD = 3$, and $CD = 2$, then the length of HD is

(A) $\frac{\sqrt{2}}{2}$ **(B)** $\frac{3}{2}$ **(C)** $\sqrt{5}$ **(D)** $\frac{5}{2}$ **(E)** NA



SOLUTION. Let CH intersect AB at F. CF is an altitude of the triangle. Since $AD = 4$, $BD = 3$, we have $AB = 5$. Since $BC = 5$, the triangles ABD and CBF are congruent (AAS). Therefore, $AF = 2$ and $FB = 3$. Let $HD = x$ so that $AH = 4 - x$. Applying Menelaus' theorem to triangle ADB and transversal CHF, we have $\frac{4-x}{x} \cdot \frac{2}{-5} \cdot \frac{3}{2} = -1$, from which $x = \frac{3}{2}$.

The correct solution is **B.** ■