MATH DAY 2009 at FAU

Competition B-Teams

THE QUESTIONS

1. What are the two rightmost digits of the number 2009^{2009}?

   (A) 09 **(B)** 49 **(C)** 61 **(D)** 89 **(E)** NA

2. The largest positive integer n such that $n+5$ divides n^3+5

   (A) is a multiple of 10 **(B)** satisfies $25 < n < 75$ **(C)** satisfies $75 < n < 120$
   (D) satisfies $120 < n < 375$ **(E)** satisfies $375 < n < 7505$

3. How many integers n such that $100 < n < 10,00$ are divisible by 2 but not by 6?

   Write your answer directly onto the answer sheet.

4. Suppose that $r_1, r_2, \ldots, r_{100}$ are the one hundred (possibly complex) roots of the equation $x^{100} - 4x + 5 = 0$. (You may assume all the roots are distinct). Compute the sum

   $$r_1^{100} + r_2^{100} + \cdot + r_{100}^{100}$$

   of the 100th powers of all the roots.

   (A) -99 **(B)** 500 **(C)** -599 **(D)** 599 **(E)** -500

5. If x is a real number, then $[x]$ denotes the largest integer less than or equal to x. For example, $[5] = 5$, $[\sqrt{2}] = 1$, $[-3] = -3$, $[-2.5] = -3$. Which of the following gives an **integer** that is **always** closest to x:

   (A) $-\left[-x + \frac{1}{2}\right]$ **(B)** $2\left[\frac{x+1}{2}\right]$ **(C)** $[x] + \frac{1}{2}$ **(D)** $-[-x]$ **(E)** NA

6. Suppose that $0 < b < a$ and the sum $a+b$, the product ab, and the difference of squares $a^2 - b^2$ are all equal. Determine a.

   (A) 2 **(B)** $\sqrt{5}$ **(C)** $\frac{\sqrt{5}+1}{2}$ **(D)** $\frac{\sqrt{5}+3}{2}$ **(E)** NA

7. The equation

   $$x^3 + ax^2 + bx + c = 0$$

   has the roots $x = 1$, $x = 2$, and $x = 3$. Determine b.

   Write your answer directly onto the answer sheet.

8. Compute the sum

   $$\binom{2009}{1}\sin 60^\circ + \binom{2009}{2}\sin 120^\circ + \binom{2009}{3}\sin 180^\circ + \cdots + \binom{2009}{2009}\sin(2009 \times 60^\circ).$$

   (A) $-(\sqrt{3})^{2009}$ **(B)** $-\frac{1}{2}(\sqrt{3})^{2009}$ **(C)** $\frac{1}{2}(\sqrt{3})^{2009}$ **(D)** $(\sqrt{3})^{2009}$ **(E)** 0 **(F)** NA

Note: If m, n are non-negative integers, $m \leq n$, then

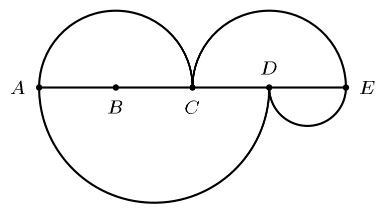
$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{m!}.$$

In this context one interprets $0! = 1$ so that $\binom{n}{0} = \binom{n}{n} = 1$. One can also define $\binom{n}{m}$ as being the m-th entry of the n-th row of the Pascal triangle, the count beginning with 0.

9. $ABCD$ is a rectangle in which the shorter side AD has length 1. The perpendiculars from B and D to the diagonal AC divide the diagonal into three equal parts. Find the length of AB.

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** $\sqrt{5}$ **(D)** 3 **(E)** NA

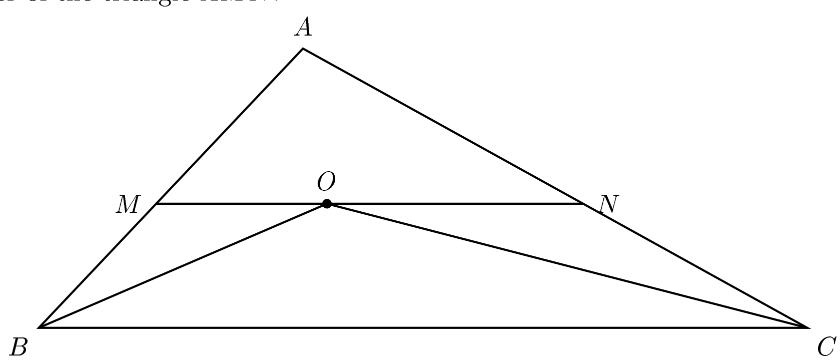
10. A line AE is divided into four equal parts by the points B, C, D. Semicircles are drawn with segments AC CE, AD and DE as diameters.



The ratio of the area enclosed above the line AE to the area enclosed below the line is

(A) $4:5$ **(B)** $5:4$ **(C)** $1:1$ **(D)** $8:9$ **(E)** NA

11. In the picture below, BO bisects the angle $\angle ABC$ and CO bisects $\angle ACB$. If $|AB| = 4$, $|BC| = 8$, and $AC = 6$, determine the perimeter of the triangle AMN.



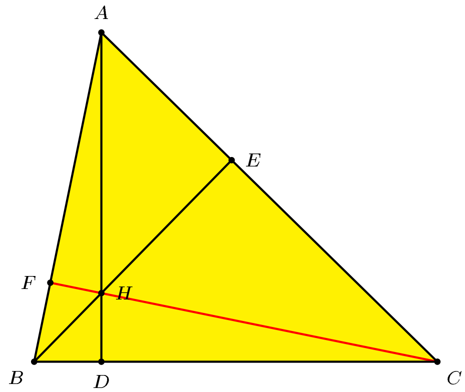
(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** NA

12. Which of the following conditions does **NOT** guarantee that the convex quadrilateral $ABCD$ is a parallelogram?

(A) $AB = CD$ and $AD = BC$ **(B)** $\angle A = \angle C$ and $\angle B = \angle D$ **(C)** $AB = CD$ and $\angle A = \angle C$
(D) $AB = CD$ and $AB // CD$ **(E)** NA

13. In triangle ABC, the altitude from A to BC meets BC at D, and the altitude from B to CA meets AD at H. If $AD = 4$, $BD = 3$, and $CD = 2$, then the length of HD is

(A) $\frac{\sqrt{2}}{2}$ **(B)** $\frac{3}{2}$ **(C)** $\sqrt{5}$ **(D)** $\frac{5}{2}$ **(E)** NA


A
E
F
H
B
D
C