On Jacobsthal Binary Sequences

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Abstract. Let \( \Sigma = \{0, 1\} \) be the binary alphabet, and \( A = \{0, 01, 11\} \) the set of three strings 0, 01, 11 over \( \Sigma \). Let \( A^* \) denote the Kleene closure of \( A \), \( \mathbb{Z}^0 \) the set of nonnegative integers, and \( \mathbb{Z}^+ \) the set of positive integers. A sequence in \( A^* \) is called a Jacobsthal binary sequence. Let \( J(n) \) denote the set of Jacobsthal binary sequences of length \( n \). For \( k \in \mathbb{Z}^+ \), \( \{s_1, s_2, \ldots, s_k\} \subseteq \mathbb{Z}^0 \) and \( n - 1 \geq s_1 > s_2 > \ldots > s_k \geq 0 \), let \( J_1(n; s_1, s_2, \ldots, s_k) \) denote the subset \( J_1(n; s_1, s_2, \ldots, s_k) = \{a_{n-1}a_{n-2} \ldots a_0 \in J(n) : a_{s_i} = 1 (1 \leq i \leq k)\} \), of \( J(n) \), and let \( N_1(n; s_1, s_2, \ldots, s_k) = |J_1(n; s_1, s_2, \ldots, s_k)| \).

When \( k = 1 \), a formula for \( N_1(n; s) \) has been derived recently. In this paper we consider the general case of \( N_1(n; s_1, s_2, \ldots, s_k) \), and study some other special types of Jacobsthal binary sequences. Some identities involving these numbers are also given.

Keywords. Jacobsthal numbers, combinatorial identities, combinatorial enumeration

Introduction

Let \( \Sigma = \{0, 1\} \) be the binary alphabet, and \( A = \{0, 01, 11\} \) the set of three strings 0, 01, 11 over \( \Sigma \). Let \( A^* \) denote the Kleene closure of \( A \), \( \mathbb{Z}^0 \) the set of nonnegative integers, and \( \mathbb{Z}^+ \) the set of positive integers. A sequence in \( A^* \) is called a Jacobsthal binary sequence. Let \( J(n) \) denote the set of Jacobsthal binary sequences of length \( n \) and let \( |J(n)| \) denote the cardinality of \( J(n) \).

The Jacobsthal numbers are defined by the recursion

\[
J_n = J_{n-1} + 2J_{n-2}, \quad n > 2
\]

(1)

together with the initial values

\[
J_0 = J_1 = 1.
\]

(2)

Note that some other authors use the initial values \( J_0 = 0, \ J_1 = 1 \) instead. Using the initial values in (2), a known result can be stated more conveniently as
\(|J(n)| = J_n. (3)\)

\(J_n\) is also called the \(n^{th}\) Jacobsthal number. For convenience, we also define

\[J_m = 0, \forall m \in \mathbb{Z}, \; m < 0. (4)\]

Based on (4), we state an obvious fact and a known result as a lemma for easy reference.

**Lemma 1** The recursion (1) can be extended as

\[J_t = J_{t-1} + 2J_{t-2}, \quad t \in \mathbb{Z}, \; t \neq 0.\]

The value of \(J_n\) (\(n \in \mathbb{Z}^0\)) can be computed by

\[J_n = \frac{1}{3}(2^{n+1} + (-1)^n), \quad n \in \mathbb{Z}^0. (5)\]

The Jacobsthal numbers have applications in such areas as tiling, graph matching, alternating sign matrices, etc. ([1,2,4,5]).

Let \(k \in \mathbb{Z}^+, \{s_1, s_2, \ldots, s_{k-1}, s_k\} \subset \mathbb{Z}^0; n - 1 \geq s_1 > s_2 > \ldots > s_k \geq 0. (6)\)

Let \(J_1(n; s_1, s_2, \ldots, s_k)\) denote the following subset of \(J(n)\):

\[J_1(n; s_1, s_2, \ldots, s_k) = \{a_{n-1}a_{n-2}\ldots a_1a_0 \in J(n) : a_{s_i} = 1 \; (1 \leq i \leq k)\},\]

i.e., the subset of Jacobsthal binary sequences that have the digit 1 at each of the \(s_i^{th}\) \((1 \leq i \leq k)\) positions from the right. Let \(N_1(n; s_1, s_2, \ldots, s_k) = |J_1(n; s_1, s_2, \ldots, s_k)|. R. Grimaldi[4] considers the case where \(k = 1\), establishing a recursion for \(N_1(n; s_1)\) and then deriving the following formula:

\[N_1(n; s) = \frac{1}{3}(2^n + (-1)^n + (-1)^{n-s}2^s) (7)\]

\[= J_n - \frac{2^s}{3}(2^{n-s} + (-1)^{n-s-1}). (8)\]

For the general case, finding a formula for \(N_1(n; s_1, s_2, \ldots, s_k)\) by using a recursion seems extremely difficult. In this article we employ a different approach to dealing with this problem, namely, considering the following dual problem of \(N_1(n; s_1, s_2, \ldots, s_k)\).

Let \(r \in \mathbb{Z}^+, \{t_1, t_2, \ldots, t_{r-1}, t_r\} \subset \mathbb{Z}^0, \; n - 1 \geq t_1 > t_2 > \ldots > t_r \geq 0. (9)\)

Let \(J_0(n; t_1, t_2, \ldots, t_r)\) denote the following subset of \(J(n)\):
\[ J_0(n; t_1, t_2, \ldots, t_r) = \{a_{n-1}a_{n-2} \ldots a_1a_0 \in J(n) : a_i = 0 \ (1 \leq i \leq r) \}, \]

i.e., the subset of Jacobsthal binary sequences that have the digit 0 at each of the \( t_i^{th} \) (1 \( \leq i \leq r \)) positions from the right. Let \( N_0(n; t_1, t_2, \ldots, t_r) = |J_0(n; t_1, t_2, \ldots, t_r)|. \)

In the next section we present characterizations of the sets \( J(n) \) and \( J_0(n; t_1, t_2, \ldots, t_r) \). Based on them, some combinatorial identities involving \( J_n, \ N_0(n; t_1, t_2, \ldots, t_r) \) and \( N_1(n; s_1, s_2, \ldots, s_k) \) are derived in Section 3. From these identities, formulas for \( N_0(n; t_1, t_2, \ldots, t_r) \) and \( N_1(n; s_1, s_2, \ldots, s_k) \) are obtained in the last section.

1. Characterizations of the sets \( J(n) \) and \( J_0(n; t_1, t_2, \ldots, t_r) \)

For easy reference we state a trivial fact, that is

**Lemma 2** For any \( i, j \in \mathbb{Z}^+ \), \( J(i)||J(j) \subseteq J(i + j) \), where \( J(i)||J(j) = \{a||b : a \in J(i), b \in J(j) \} \) and \( || \) stands for the concatenation operation on strings.

We now characterize the set \( J(n) \). We need

**Lemma 3** Let \( l \in \mathbb{Z}^+ \). The string \( \alpha \) of the 0-digit followed by \( l-1 \) 1-digits is a Jacobsthal binary string of length \( l \).

**Proof.** If \( l = 2m + 1 \) for some \( m \in \mathbb{Z}^0 \), the \( l - 1 = 2m \) 1-digits in \( \alpha \) can be regarded as \( m \) copies of the string 11. Since both strings 11, 0 \( \in A \), we know \( \alpha \in A \). If \( l = 2m \) for some \( m \in \mathbb{Z}^0 \), the last \( l - 2 = 2m - 2 \) 1-digits in \( \alpha \) can be regarded as \( m - 1 \) copies of the string 11. Since both string 11, 01 \( \in A \), we know \( \alpha \in A \). \( \square \)

**Theorem 1** For any \( n \in \mathbb{Z}^+ \), a binary sequence of length \( n \) is in \( J(n) \) if and only if it is an all-1 sequence of even length or its first 0-digit from the left is preceded by an all-1 subsequence of even length.

**Proof.** Since the string 1 \( \notin A \) but the string 11 \( \in A \), the all-1 sequence of length \( n \) is in \( J(n) \) if and only if \( n \) is even. Therefore, in what follows we only need to consider the case in which the sequence \( a_{n-1}a_{n-2} \ldots a_1a_0 \) has at least one 0-digit.

Let \( a_{n-1} \) be the first 0-digit from the left. Then

\[ a_{n-1} = a_{n-2} = \ldots = a_{n-(i-1)} = 1. \]

Since the two strings 1, 10 \( \notin A \), in order for \( a_{n-1}a_{n-2} \ldots a_1a_0 \) to be in \( J(n) \), the subsequence \( a_{n-1}a_{n-2} \ldots a_{n-(i-1)} \) has to be formed by copies of the element 11 \( \in A \). This is impossible when \( i - 1 \) is odd.

We now prove that when \( i - 1 \) is even, the sequence \( a_{n-1}a_{n-2} \ldots a_1a_0 \) is in \( J(n) \) by induction on the number, say \( u \), of 0-digits in the sequence. For the case where \( u = 1 \), let \( a_i = 0 \). By Lemma 3, the subsequence \( a_ia_{i-1} \ldots a_1a_0 \in J(i+1) \). Recalling that
Theorem 2
By Theorem 1 we can give a characterization of the set \( J_n \). When \( n \) is a partition of \( J_n \), we know \( a_{n-1}a_{n-2} \ldots a_{i+1} \in J(n-i-1) \) we know \( a_{n-1}a_{n-2} \ldots a_i a_0 \in J(n) \) by Lemma 2. This establishes the induction basis.

For the inductive step, suppose that \( u > 1 \) and the conclusion is true for any sequence having exactly \( u-1 \) 0-digits. Let \( a_l \) be the first 0-digit from the right in a sequence having \( u \) 0-digits. By Lemma 3, we know \( a_la_{l-1} \ldots a_0 = 011 \ldots 1a_0 \in J(l+1) \). By the induction hypothesis, \( a_{n-1}a_{n-2} \ldots a_{l+1} \in J(n-l-1) \). Therefore, \( a_{n-1}a_{n-2} \ldots a_l a_0 \in J(n) \) by Lemma 2. This completes the induction. \( \Box \)

From this theorem, one can obtain the known formula (5) for \( |J(n)| \).

Corollary 1

\[
|J(n)| = \frac{2^{n+1} + (-1)^n}{3},
\]

Proof. Let \( J(n, i) \) denote the set of such Jacobsthal binary sequences that have their first 0-digit at the \((2i+1)^{st}\) position from the left, and \( \Delta_n \) the set consisting of the all-1 sequence of length \( n \) when \( 2 \mid n \), and \( \Delta_n = \emptyset \) when \( 2 \nmid n \). Then

\[
J(n) = ( \bigcup_{0 \leq i \leq (n-1)/2} J(n, i) ) \cup \Delta_n
\]

is a partition of \( J(n) \). By Theorem 1, when \( n = 2m \ (m \in \mathbb{Z}^+) \), we have :

\[
|J(n)| = \sum_{i=0}^{m-1} 2^{2m-(2i+1)} + 1 = \frac{1}{2} \sum_{i=0}^{m-1} 4^{m-i} + 1 = \frac{1}{2} \sum_{i=1}^{m} 4^i + 1 = 2 \sum_{i=0}^{m-1} 4^i + 1 = 2(\frac{4^m-1}{3}) + 1 = \frac{2^{m+1} + (-1)^n}{3}.
\]

When \( n = 2m + 1 \ (m \in \mathbb{Z}^0) \), we have :

\[
|J(n)| = \sum_{i=0}^{m} 2^{2m+1-(2i+1)} = \sum_{i=0}^{m} 2^{2(m-i)} = \sum_{i=0}^{m} 2^{2i} = \sum_{i=0}^{m} 4^i = \frac{4^{m+1}-1}{3} = \frac{2^{m+1} + (-1)^n}{3}. \Box
\]

By Theorem 1 we can give a characterization of the set \( J_0(n; t_1, t_2, \ldots, t_r) \). Recall that the parameters satisfy (9):

\[
r \in \mathbb{Z}^+, \{t_1, t_2, \ldots, t_{r-1}, t_r\} \subset \mathbb{Z}^0, n - 1 > t_1 > t_2 > \ldots > t_r \geq 0.
\]

Theorem 2 For any \( n \in \mathbb{Z}^+ \), the binary sequence \( a_{n-1}a_{n-2} \ldots a_1 a_0 \) of length \( n \) is in \( J_0(n; t_1, t_2, \ldots, t_r) \) if and only if the subsequence \( a_{n-1}a_{n-2} \ldots a_{t_i + 1} \) is in \( J(n-1-t_1) \) and \( a_{t_i} = 0 \ (1 \leq i \leq r) \).
Proof. Let $a_j$ be the first 0-digit from the left. Then $j \geq t_1$. By Theorem 1, $a_{n-1}a_{n-2} \ldots a_1 a_0 \in J(n)$ if and only if the entries before $a_j$ are all 1’s, i.e., $2|n - 1 - j$, which is the necessary and sufficient condition for $a_{n-1}a_{n-2} \ldots a_{t_1 + 1}$ to be in $J(n - 1 - t_1)$. □

It is somewhat surprising that whether $a_{n-1}a_{n-2} \ldots a_1 a_0 \in J_0(n; t_1, t_2, \ldots, t_r)$ or not is determined only by the subsequence $a_{n-1}a_{n-2} \ldots a_{t_1 + 1}$ and $a_{t_i} = 0$ $(1 \leq i \leq r)$, but is independent of the digits $a_j$ $(0 \leq j \leq t_1 - 1, j \neq t_i)$.

Based on these theorems, some combinatorial identities involving $J_n$, $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ can be established, which will be presented in the next section.

2. Some Combinatorial Identities Involving $J_n$, $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$

In this section some combinatorial identities involving $J_n$, $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ are proved. Applying them to obtain formulas for $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ will be the task of the next section.

We need a simple lemma:

Lemma 4 For any $n \in \mathbb{Z}^0$,  
\[ 2^n = 3J_{n-1} + (-1)^n. \]

Proof. Recalling that $J_{-1} = 0$ (cf. (4)), we know that the statement is true when $n = 0$. When $n \in \mathbb{Z}^+$, the statement is equivalent to (5). □

We can now state the following

Theorem 3

\[
N_0(n; t_1, t_2, \ldots, t_r) = [3J_{t_1 - r} + (-1)^{t_1 - r + 1}]J_{n - t_1 - 1}
\]

\[
N_0(n; t_1, t_2, \ldots, t_r) = J_{n-r} + (-1)^{n-r-1}J_{t_1 - r}
\]

Proof. By Theorem 2, for a sequence $a_{n-1}a_{n-2} \ldots a_1 a_0$ in $J_0(n; t_1, t_2, \ldots, t_r)$, there are $|J(n - t_1 - 1)| = J_{n-t_1-1}$ many choices for the subsequences $a_{n-1}a_{n-2} \ldots a_{t_1 + 1}$. For each of these choices, there are two choices for each of the digits $a_j$ $(0 \leq j \leq t_1 - 1, j \neq t_2, t_3, \ldots, t_r)$. Noting that $a_{t_j} = 0$ $(1 \leq j \leq r)$, we have

\[
N_0(n; t_1, t_2, \ldots, t_r) = |J(n - t_1 - 1)| \cdot 2^{t_1 - r + 1}
\]

\[
= J_{n-t_1-1}2^{t_1 - r + 1}.
\]

By Lemma 4,

\[ 2^{t_1 - r + 1} = 3J_{t_1 - r} + (-1)^{t_1 - r + 1}. \]
Therefore,

\[ N_0(n; t_1, t_2, \ldots, t_r) = J_{n-t_1-1} [3J_{t_1-r} + (-1)^{t_1-r+1}], \]

which is (10). Similarly, we can also write

\[ N_0(n; t_1, t_2, \ldots, t_r) = \]
\[ = J_{n-t_1-1} 2^{t_1-r+1} \]
\[ = \frac{1}{3} [2^{n-t_1} J_{n-t_1-1} 2^{t_1-r+1}] \]
\[ = \frac{1}{3} [2^{n-r+1} J_{n-r} + (-1)^{n-t_1-1} [3J_{t_1-r} + (-1)^{t_1-r+1}]] \]
\[ = J_{n-r} + (-1)^{n-t_1-1} J_{t_1-r}, \]

which proves (11). □

From this theorem, an identity can be immediately derived.

**Corollary 2** We have the identity

\[ [3J_{t_1-r} + (-1)^{t_1-r+1}] J_{n-t_1-1} = J_{n-r} + (-1)^{n-t_1-1} J_{t_1-r}. \]

This identity can also be checked by using (5).

Let us look at the cases \( r = 1 \) and \( r = 2 \).

**Corollary 3** If \( n - 1 \geq u \geq 0 \), then

\[ N_0(n; u) = [3J_{u-1} + (-1)^u] J_{n-u-1} \] (12)
\[ N_0(n; u) = J_{n-1} + (-1)^{n-u-1} J_{u-1} \] (13)

**Example 1** From (13) and \( J_0 = J_1 = 1, J_2 = 3 \), we have

\[ N_0(1; 0) = J_0 + (-1)^0 J_{-1} = 1, \]
\[ N_0(2; 0) = J_1 + (-1)^1 J_{-1} = 1, \]
\[ N_0(2; 1) = J_1 + (-1)^0 J_0 = 2, \]
\[ N_0(3; 0) = J_2 + (-1)^2 J_{-1} = 3, \]
\[ N_0(3; 1) = J_2 + (-1)^1 J_0 = 2, \]
\[ N_0(3; 2) = J_2 + (-1)^0 J_1 = 4. \]

The corresponding subsets of \( J(n) \) are

\[ J_0(1; 0) = \{0\}, J_0(2; 0) = \{00\}, J_0(2; 1) = \{00, 01\}. \]
\[ J_0(3; 0) = \{000, 010, 110\}, J_0(3; 1) = \{000, 001\}, J_0(3; 2) = \{000, 001, 010, 011\}. \]
Corollary 4. If \( n - 1 \geq u \geq 0 \), then
\[
[3J_{u-1} + (-1)^u] J_{n-u-1} = J_{n-1} + (-1)^{n-u-1} J_{u-1}.
\]

For \( N_1(n; s_1, s_2, \ldots, s_k) \), we have

Theorem 4. Suppose that \( s_1, s_2, \ldots, s_k \) satisfy (6). Then \( N_1(n; s_1, s_2, \ldots, s_k) = \)
\[
J_n + \sum_{1 \leq r \leq k} (-1)^r \sum_{1 \leq i \leq k-r+1} \binom{k-i}{r-1} J_{n-r} + (-1)^{n-s_i-1} J_{s_i-r}.
\]

Proof. First of all, for any \( 1 \leq r \leq k \), by (11) we have:
\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} N_0(n; s_{i_1}, s_{i_2}, \ldots, s_{i_r}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} J_{n-r} + (-1)^{n-s_{i_1}-1} J_{s_{i_1}-r}.
\]

Since \( 1 \leq i_1 < i_2 < \ldots < i_r \leq k \), the index \( i_1 \) must satisfy \( 1 \leq i_1 \leq k-r+1 \). After \( i_1 \) has been chosen from this range, there are \( \binom{k-i_1}{r-1} \) ways of choosing \( i_2, \ldots, i_r \). Since the summands \( J_{n-r} + (-1)^{n-s_{i_1}-1} J_{s_{i_1}-r} \) do not depend on the values of \( i_2, \ldots, i_r \), we have:
\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} J_{n-r} + (-1)^{n-s_{i_1}-1} J_{s_{i_1}-r} = \sum_{1 \leq i_1 \leq k-r+1} \binom{k-i_1}{r-1} J_{n-r} + (-1)^{n-s_{i_1}-1} J_{s_{i_1}-r}.
\]
Further, using \( i \) to substitute for \( i_1 \) in the summation on the right hand side, yields:
\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} N_0(n; s_{i_1}, s_{i_2}, \ldots, s_{i_r}) = \sum_{1 \leq i_1 \leq k-r+1} \binom{k-i}{r-1} J_{n-r} + (-1)^{n-s_i-1} J_{s_i-r}.
\]

By the inclusion-exclusion principle, \( N_1(n; s_1, s_2, \ldots, s_k) = \)
\[
J_n + \sum_{1 \leq r \leq k} (-1)^r \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} N_0(n; s_{i_1}, s_{i_2}, \ldots, s_{i_r}) = \]
\[
J_n + \sum_{1 \leq r \leq k} (-1)^r \sum_{1 \leq i_1 \leq k-r+1} \binom{k-i}{r-1} J_{n-r} + (-1)^{n-s_i-1} J_{s_i-r},
\]
which proves (4). \( \Box \)

Similarly, using (10) instead of (11) yields the following:

Theorem 5. Suppose that \( s_1, s_2, \ldots, s_k \) satisfy (6). Then \( N_1(n; s_1, s_2, \ldots, s_k) = \)
\[
J_n + \sum_{1 \leq r \leq k} (-1)^r \sum_{1 \leq i_1 \leq k-r+1} \binom{k-i}{r-1} [3J_{s_i-r} + (-1)^{s_i-r+1} J_{s_i-r}].
\]
Let us look at the cases for \( k = 1, 2. \)

**Corollary 5** For any \( n \in \mathbb{Z}^+ \) and \( n - 1 \geq u \geq 0, \)

\[
N_1(n; u) = 2J_{n-2} + (-1)^{n-u} J_{u-1} \\
N_1(n; u) = J_n - [3J_{u-1} + (-1)^u]J_{n-u-1}.
\]

Proof. By Theorem 4 and Lemma 1,

\[
N_1(n; u) = J_n + (-1)^{1 \cdot \binom{n-2}{1-u}} [J_{n-1} + (-1)^{n-u} J_{u-1}] \\
= J_n - J_{n-1} + (-1)^{n-u} J_{u-1} \\
= 2J_{n-2} + (-1)^{n-u} J_{u-1}.
\]

And by Theorem 5 we obtain:

\[
N_1(n; u) = J_n + (-1)^{1 \cdot \binom{n-2}{1-u}} [3J_{u-1} + (-1)^u J_{n-u-1}] \\
= J_n - [3J_{u-1} + (-1)^u J_{n-u-1}]. \quad \square
\]

**Example 2** By Corollary 5, we have:

\[
N_1(1; 0) = 2J_1 + J_{-1} = 0, \quad N_1(2; 0) = 2J_0 + J_{-1} = 2, \quad N_1(2; 1) = 2J_0 - J_0 = 1, \\
N_1(3; 0) = 2J_1 - J_{-1} = 2, \quad N_1(3; 1) = 2J_1 + J_0 = 3, \quad N_1(3; 2) = 2J_1 - J_1 = 1.
\]

The corresponding subsets of \( J(n) \) are

\[
J_1(1; 0) = \emptyset, \quad J_1(2; 0) = \{01, 11\}, \quad J_1(2; 1) = \{11\}, \\
J_1(3; 0) = \{001, 011\}, \quad J_1(3; 1) = \{010, 011, 110\}, \quad J_1(3; 2) = \{110\}.
\]

**Example 3** Applying Corollary 5, we have

\[
N_1(1; 0) = J_1 - [3J_{-1} + 1]J_0 = 1 - 1 = 0. \\
N_1(2; 0) = J_2 - [3J_{-1} + 1]J_1 = 3 - 1 = 2. \\
N_1(2; 1) = J_2 - [3J_0 - 1]J_0 = 3 - 2 = 1. \\
N_1(3; 0) = J_3 - [3J_{-1} + 1]J_2 = 5 - 3 = 2. \\
N_1(3; 1) = J_3 - [3J_0 - 1]J_1 = 5 - 2 = 3. \\
N_1(3; 2) = J_3 - [3J_1 + 1]J_0 = 5 - 4 = 1.
\]

The corresponding subsets of \( J(n) \) have been shown in Example 2.
Now let us turn to the case of \( k = 2 \). In this case, \( n > 1 \).

**Corollary 6** For any \( n \in \mathbb{Z}^+ \), \( n \geq 2 \), and \( n - 1 \geq u > v \geq 0 \), we have:

\[
N_1(n; u, v) = 2[J_{n-2} - J_{n-3}] + (-1)^{n-u}[J_{u-1} - J_{u-2}] + (-1)^{n-v}J_{v-1}.
\]  

(14)

For any \( n \in \mathbb{Z}^+ \), \( n \geq 3 \), \( n - 1 \geq u > v \geq 0 \), \( u \geq 2 \), we have:

\[
N_1(n; u, v) = 4J_{n-4} + (-1)^{n-u}2J_{u-3} + (-1)^{n-v}J_{v-1}.
\]  

(15)

Proof. By Theorem 4, \( N_1(n; s_1, s_2) = \)

\[
J_n + (-1)^1 \sum_{1 \leq i \leq 2} \binom{2 - 1}{i - 1} [J_{n-i} + (-1)^{n-s_i-1}J_{s_i-1}] + \\
+ \binom{2 - 1}{2 - 1} [J_{n-2} + (-1)^{n-s_1-1}J_{s_1-2}] =
\]

\[
J_n - [J_{n-1} + (-1)^{n-s_1-1}J_{s_1-1} + J_{n-1} + (-1)^{n-s_2-1}J_{s_2-1}] + \\
+ [J_{n-2} + (-1)^{n-s_1-1}J_{s_1-2}] =
\]

\[
J_n - 2J_{n-1} + J_{n-2} + (-1)^{n-s_1}J_{s_1-1} + (-1)^{n-s_2}J_{s_2-1} + \\
+ (-1)^{n-s_1-1}J_{s_1-2} =
\]

\[
2[J_{n-2} - J_{n-3}] + (-1)^{n-s_1}[J_{s_1-1} - J_{s_1-2}] + (-1)^{n-s_2}J_{s_2-1}.
\]

Substituting \( u, v \) for \( s_1, s_2 \), respectively, gives (14).

When \( n \geq 3 \), and \( s_1 \geq 2 \), by Lemma 1 we have:

\[
J_{n-2} - J_{n-3} = 2J_{n-4}, \quad J_{s_1-1} - J_{s_1-2} = 2J_{s_1-3}.
\]

So :

\[
N_1(n; s_1, s_2) = 2[J_{n-2} - J_{n-3}] + (-1)^{n-s_1}[J_{s_1-1} - J_{s_1-2}] + (-1)^{n-s_2}J_{s_2-1} =
\]

\[
4J_{n-4} + (-1)^{n-s_1}2J_{s_1-3} + (-1)^{n-s_2}J_{s_2-1}.
\]

Substituting \( u, v \) for \( s_1, s_2 \), respectively, gives (15). \( \square \)

The identities in this section can be used to give formulas for \( N_0(n; t_1, t_2, \ldots, t_r) \) and \( N_1(n; s_1, s_2, \ldots, s_k) \), which will be presented in the next section.

3. Formulas for \( N_0(n; t_1, t_2, \ldots, t_r) \) and \( N_1(n; s_1, s_2, \ldots, s_k) \)

For \( N_0(n; t_1, t_2, \ldots, t_r) \), we have:
Theorem 6  The following holds :

\[ N_0(n; t_1, t_2, \ldots, t_r) = \left( \frac{1}{3} \right)^{t_1+1-r} [2^{n-t_1} + (-1)^{n-t_1-1}] \]  \hspace{1cm} (16)

Proof. From the proof of Theorem 3 and equality (5), we have

\[ N_0(n; t_1, t_2, \ldots, t_r) = J_{n-1-t_1} \cdot 2^{t_1+1-r} \]

\[ = \frac{1}{3} 2^{t_1+1-r} [2^{n-t_1} + (-1)^{n-t_1-1}] . \]

Note that \( N_0(n; t_1, t_2, \ldots, t_r) \) only depends on the parameters \( n, t_1 \) and \( r \), and is independent of the values of the parameters \( t_2, \ldots, t_r \).

Theorems 3 and 4 provide an explicit formulas for \( N_1(n; s_1, s_2, \ldots, s_k) \), as shown in the following theorem. Its proof is obvious and will be omitted.

Theorem 7  Suppose that \( s_1, s_2, \ldots, s_k \) satisfy (6). Then

\[ N_1(n; s_1, s_2, \ldots, s_k) = \left( \frac{1}{3} \right)(2^{n+1} + (-1)^n) + \]

\[ + \left( \frac{1}{6} \right) \sum_{1 \leq r \leq k} \sum_{1 \leq i \leq k-r+1} (-1)^{k-r+1} 2^{s_i-r+1} \] \( (r-1)^2 \) \( s_i \)

\[ \frac{1}{6} \sum_{1 \leq r \leq k} (-1)^r 2^{s_i-r+1} + (-1)^{n-s_i-1} . \]

When \( k = 1 \), we have :

Corollary 7

\[ N_1(n; s) = \frac{1}{3} (2^{n+1} - 2^s [2^{n-s} + (-1)^{n-s-1}] + (-1)^n) . \]  \hspace{1cm} (17)

Example 4  By (17), the first several values of \( N_1(n; s) \) can be computed as follows.

\[ N_1(1; 0) = \frac{1}{3} (2^2 - 2^0 [2^2 + (-1)^1] + (-1)^1) = 0, \]

\[ N_1(2; 0) = \frac{1}{3} (2^3 - 2^0 [2^2 + (-1)^1] + (-1)^2) = 2, \]

\[ N_1(2; 1) = \frac{1}{3} (2^3 - 2^1 [2^1 + (-1)^0] + (-1)^2) = 1, \]

\[ N_1(3; 0) = \frac{1}{4} (2^4 - 2^0 [2^3 + (-1)^2] + (-1)^3) = 2, \]

\[ N_1(3; 1) = \frac{1}{4} (2^4 - 2^1 [2^2 + (-1)^1] + (-1)^3) = 3, \]

\[ N_1(3; 2) = \frac{1}{4} (2^4 - 2^2 [2^1 + (-1)^0] + (-1)^3) = 1. \]

The corresponding subsets of \( J(n) \) have been shown in Example 2.

When \( k = 2 \), we have :
Corollary 8  For any $n \geq 2$ and $n - 1 \geq u > v \geq 0$, we have:

$$N_1(n; u, v) = \left(\frac{1}{3}\right) [2^{n-1} + (-1)^{n-u}2^{u-1} + (-1)^{n-v}2^{v} + (-1)^{n}].$$

References

[5]  H. Silvia, Tiling an $m$-by-$n$ Area with Squares of Size up to $k$-by-$k$ ($m \leq 5$), Congressus Numerantium, 140(1999), 43-64.