Reconstructing a VW plane from its collineation group

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Abstract

In this study we analyze the structure of the full collineation group of certain Veblen-Wedderburn(VW) planes of orders 5^2, 7^2 and 11^2. We also discuss a reconstruction method using their collineation groups.

1 Introduction

In [1] some group–theoretical methods for constructing both the Hughes plane of order q^2 and the Figueroa plane of order q^3, q an odd prime power, are discussed. The method is using the well–known linear group $GL(3, q)$. In this paper, we discuss a reconstruction method for certain non–desarguesian VW planes of some particular orders from their collineation groups.

In section 2 we introduce some notation, definitions and preliminaries. In section 3 we discuss a particular *Planar Ternary Ring* (R, T) of order p^2, p an odd prime, which gives rise to the non–desarguesian VW planes α, β and γ, of orders 5^2, 7^2 and 11^2, respectively. In section 4 we analyze the structure of the full collineation groups of α, β and γ. In Section 5 we discuss how to reconstruct these particular VW planes from their collineation groups.

2 Preliminaries

We assume the reader is familiar with the basics of finite projective planes and group theory. If G is a group acting on the set X, we denote by $G|X$ the group action of G on X. If π is a projective plane, we denote by P_π and L_π, the sets of *points* and *lines* of π, respectively. We denote by (a) the set of points incident with $a \in L_\pi$ and by (A) the set of lines incident with $A \in P_\pi$. If $A, B \in P_\pi$ and $A \neq B$, we denote by AB the line in L_π incident with A and B. Symmetrically if $a, b \in L_\pi$, $a \neq b$, ab denotes the point in P_π incident with a and b. By a *quadrangle* of a plane π we mean a set of four points no three of which are collinear. A *collineation* of a projective plane π of order n is a permutation of its points which maps lines onto lines [2]. The set of all collineations of π forms a group under composition, called the *full collineation group* G_π of π.

Veblen-Wedderburn (VW) systems are algebraic systems used to coordinatize projective planes, and planes coordinatized by VW systems are called VW planes. A VW system $(R, +, \cdot)$ of elements with operations $+$ and $\cdot$ satisfies the following axioms:

(i) $(R, +)$ is commutative.

(ii) $(R \setminus \{0\}, \cdot)$ is a loop.

(iii) $(a + b)c = ac + bc$, $a, b, c \in R$.

(iv) If $a \neq b$, $xa = xb + c$ has a unique solution x.

See [4] for further information about VW systems.

A *Planar Ternary Ring* (PTR) is a structure (R, T), where R is a nonempty set containing distinct elements called 0 and 1, and $T : R^3 \to R$ satisfying the following:

(i) $T(a, 0, b) = T(0, a, b) = b$, $\forall a, b \in R$.

(ii) $T(1, a, 0) = T(a, 1, 0) = a$, $\forall a \in R$.

(iii) For every $a, b, c \in R$, $T(a, b, x) = c$ has a unique solution $x \in R$.

(iv) For every $a, b, c, d \in R$, where $a \neq c$, $T(x, a, b) = T(x, c, d)$ has a unique solution $x \in R$.

(v) For every $a, b, c, d \in R$, where $a \neq c$, each of $T(a, x, y) = b$ and $T(c, x, y) = d$ has a unique solution $(x, y) \in R^2$.

Note that the fifth axiom is redundant if R is finite. For further information about PTR's see [3].

Given a certain PTR, the corresponding projective plane π, with points P_π, lines L_π and incidence $\mathrm{I} \subset P_\pi \times L_\pi$, is constructed as follows:

(i) $P_\pi = \{(x,y) : x, y \in R\} \cup \{(x) : x \in R\} \cup \{(\infty)\}$,

(ii) $L_\pi = \{[a,b] : a, b \in R\} \cup \{[a] : a \in R\} \cup \{[\infty]\}$,

(iii) For all $a, b, x, y \in R$, $(x,y)\ \mathrm{I}\ [a,b]$ if and only if $T(a,x,y) = b$,

(iv) $(x,y)\ \mathrm{I}\ [a]$, $(x)\ \mathrm{I}\ [a,b]$ if and only if $x = a$,

(v) $(x)\ \mathrm{I}\ [\infty]$, $(\infty)\ \mathrm{I}\ [a]$, $(\infty)\ \mathrm{I}\ [\infty]$,

(vi) $(x,y) \not\mathrm{I}\ [\infty]$, $(x) \not\mathrm{I}\ [a]$, $(\infty) \not\mathrm{I}\ [a,b]$.

3 A VW plane

Let $\mathbb{F}$ be a finite field of order p^2, p an odd prime, and R the set of elements of $\mathbb{F}$. Define $T : R^3 \to R$ as follows: $T(a,b,c) = ab + c$ if b is a square in $\mathbb{F}$, and $T(a,b,c) = a^p b + c$ if b is not a square in $\mathbb{F}$.

Proposition 1 *Let R and T be as described above. Then (R,T) is a PTR.*

Proof: Let $a, b, c \in R$ and a be a square in R. Then $T(a,0,b) = a0 + b = b = 0a + b = T(0,a,b)$, $T(a,1,0) = a1 + 0 = a = 1a + 0 = T(1,a,0)$, and $T(b,a,x) = ba + x = c$ has a unique solution $x \in R$. If a is not a square in R, then $T(a,0,b) = a0 + b = b = 0^p a + b = T(0,a,b)$, $T(a,1,0) = a1 + 0 = a = 1^p a + 0 = T(1,a,0)$, and $T(b,a,x) = b^p a + x = c$ has also a unique solution $x \in R$.
Now, let $a, b, c, d \in R$, where $a \neq c$ and $a, c \neq 0$. We have the following cases:

(i) If a and c are both squares in R, then $T(x,a,b) = T(x,c,d) \Leftrightarrow xa + b = xc + d$ and $xa + b = xc + d$ has a unique solution $x \in R$.

(ii) If a is not a square and c is a square, then $T(x,a,b) = T(x,c,d) \Leftrightarrow x^p a + b = xc + d$. This equation has a unique solution $x \in R$. See [5] for a proof.

(iii) If neither a nor c is a square in R, then $T(x,a,b) = T(x,c,d) \Leftrightarrow x^p a + b = x^p c + d \Leftrightarrow x^p = (v/u)$, where $u = a - c \neq 0$ and $v = d - b$. But there exists $t' \in R$ such that $(t')^p = (v/u)$. Therefore, $x^p = (t')^p$. Hence there is a unique solution for $T(x,a,b) = T(x,c,d)$.

Hence, (R,T) is a PTR. □

In this study we use this particular *Planar Ternary Ring* (R,T) of order p^2, $p = 5, 7$ or 11, to construct the non-desarguesian projective planes α, β, and γ of orders 5^2, 7^2 and 11^2, respectively. It follows easily from the definition that α, β, and γ are VW planes.

We compute the full collineation groups G_α, G_β and G_γ of the planes α, β, and γ, respectively. Then we ask the following question: "Is it possible to reconstruct the planes α, β, and γ by only using their collineation groups?".

4 Structure of G_π

Let π be one of the planes α, β, or γ. Since π is of order p^2, $p = 5, 7$ or 11, we assume that $P_\pi = \{A_0, A_1, ..., A_{p^4+p^2}\}$ and $L_\pi = \{a_0, a_1, ..., a_{p^4+p^2}\}$ throughout the article. We observe that G_π is not transitive on points and lines. Furthermore, there are three orbits on points, namely Θ_1, Θ_2 and Θ_3, of lengths 1, $2p^2$ and $p^4 - p^2$, and three orbits on lines, namely Γ_1, Γ_2 and Γ_3, of lengths 2, $p^2 - 1$ and p^4, respectively. Let $\Gamma_1 = \{a_0, a_1\}$, where $(a_0) = \{A_0, A_1, ..., A_{p^2}\}$ and $(a_1) = \{A_0, A_{p^2+1}, ..., A_{2p^2}\}$. Then we have that $\Gamma_2 = (A_0) \setminus \Gamma_1$ and $\Gamma_3 = L_\pi \setminus (A_0)$. Moreover, $\Theta_1 = \{A_0\}$, $\Theta_2 = ((a_0) \cup (a_1)) \setminus \{A_0\}$ and $\Theta_3 = P_\pi \setminus ((a_0) \cup (a_1))$. Furthermore, the actions $G_\pi \mid \Theta_2$ and $G_\pi \mid \Theta_3$ are faithful.

There is a subgroup $K \leq G_\pi$, of order $p^2\ (p^2 - 1)$, and K is normal in a subgroup $H < G_\pi$, where $[H : K] = 2$. See Figure 1. Furthermore, there is a cyclic subgroup $C < K$ of order $(p^2 - 1)/2$. If $C = \langle x \rangle$, then there is an element $y \in K$ such that $y^2 x^{(p^2-1)/4} = 1_{G_\pi}$ if $p \equiv 3\ (4)$, and $y^2 x^{(p^2-1)/8} = 1_{G_\pi}$ if $p \equiv 1\ (4)$. Moreover, the Sylow p–Subgroup $Syl_p < K$ is of order p^2 and K is the split extension of Syl_p by the subgroup $\langle x, y \rangle$ generated by x and y. See the appendix for the presentations of K in G_α, G_β and G_γ. In addition, there is an involution m such that $H = \langle K, m \rangle$. The generators of the subgroup H, namely x, y, a, b and m, are represented as permutations on the subset $\{1, ..., p^2\}$. Further, there is an involution $u \in G_\pi \setminus H$ such that for $H' = u^{-1}Hu$, $H \cap H' = \langle m \rangle$ and $G_\pi = \langle H, u \rangle = \langle H, H', u \rangle$. See the appendix for the size and generators of the full collineation groups G_α, G_β and G_γ.

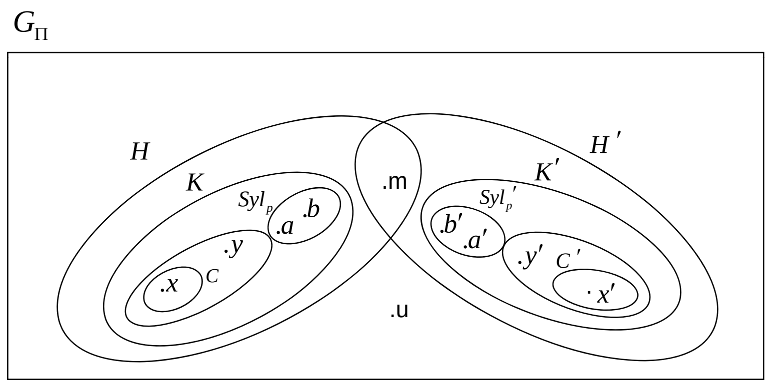

Figure 1: The full collineation group G_π

5 Reconstruction from G_π

Counting Principle. Let a_0 and a_1 (as described above) intersect each other at A_0. A point A is said to be of *type*–I if $A \in (a_0) \cup (a_1)$, and of *type*–II, otherwise. Similarly, a line a is of *type*–I if $a = AA_0$, where $A \neq A_0$, and of *type*–II, otherwise. Let $A_i \neq A_j$, $A_r \neq A_s$ be points of *type*–I, where $A_i, A_j \in (a_0) \setminus \{A_0\}$ and $A_r, A_s \in (a_1) \setminus \{A_0\}$. Then it easily follows that $Q = \{A_i, A_j, A_r, A_s\}$ is a quadrangle in π and there are $\binom{p^2}{2}\binom{p^2}{2}$ such quadrangles constructed by the points of a_0 and a_1.

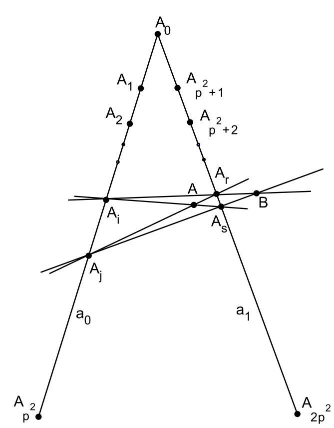

Figure 2: Counting principle

The set of intersection points of lines passing through all pairs of the points of Q is $\{A_i, A_j, A_r, A_s, A, B, A_0\}$, where A and B are distinct points of *type*–II. See Figure 2. Therefore, there are $2\binom{p^2}{2}\binom{p^2}{2}$ points of *type*–II determined by the quadrangles which are constructed as above. However, let A be any point of *type*–II, then there are $\binom{p^2}{2}$ different pairs of lines of *type*–II intersecting at A. Hence, there are $p^2(p^2-1)$ distinct points of *type*–II determined by such quadrangles. We also have that there are $2p^2+1$ distinct points of *type*–I. This leads to the following lemma.

Lemma 1 *All points of π are determined by the quadrangles as described above.*

Reconstruction. We define $S_{g,H} = \{h^{-1}gh \mid h \in H\}$ for any subgroup $H \leq G_\pi$ and $g \in G_\pi$. There is a cyclic subgroup $C' \leq K'$ of order $(p^2-1)/2$ such that $C' = u^{-1}Cu$, where $C \leq K$ is cyclic and u is the involution described above. See Figure 1. Since p is odd and C' is cyclic, C' contains exactly one involution which we call ι'.

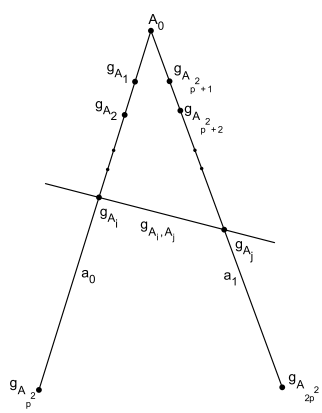

Figure 3: Representing certain points and lines by involutions

Recall that K' is the split extension of Syl'_p by the subgroup $\langle x', y'\rangle$ generated by x' and y', where $x' = u^{-1}xu$ and $y' = u^{-1}yu$. Now consider the set S_{ι',Syl'_p}. Then it easily follows that $|S_{\iota',Syl'_p}| = |Syl'_p| = p^2$ i.e. S_{ι',Syl'_p} contains exactly p^2 involutions. Our analysis of the elements in S_{ι',Syl'_p} shows the following :

(i) $\{A_{p^2+1}, \ldots, A_{2p^2}, A_0\} \subset Fix(s)$ for each $s \in S_{\iota',Syl'_p}$.

(ii) $(a_0) \cap Fix(s) = \{A_i, A_0\}$ for some i, $1 \leq i \leq p^2$, and $s \in S_{\iota',Syl'_p}$.

(iii) $(a_0) \cap Fix(s_1) \neq (a_0) \cap Fix(s_2)$ for distinct elements $s_1, s_2 \in S_{\iota', Syl'_p}$.

Therefore, there is a one–to–one correspondence between the points in $(a_0) \setminus \{A_0\}$ and the involutions in S_{ι', Syl'_p}. Moreover, we can represent the points on a_0, except A_0, by the involutions in S_{ι', Syl'_p}. Hence, we write $S_{\iota', Syl'_p} = \{g_{A_1}, \ldots, g_{A_{p^2}}\}$. Symmetrically, there is a single involution $\iota \in C$ and the points on a_1, except A_0, can be represented by the involutions in S_{ι, Syl_p}. Similarly, we write $S_{\iota, Syl_p} = \{g_{A_{p^2+1}}, \ldots, g_{A_{2p^2}}\}$. See Figure 3.

Let $g_{A_i,A_j} = g_{A_i} g_{A_j}$ for some $A_i \in (a_0) \setminus \{A_0\}$ and $A_j \in (a_1) \setminus \{A_0\}$, then $g_{A_i,A_j} \in G_\pi$ is an involution such that $Fix(g_{A_i,A_j}) \cap ((a_0) \cup (a_1)) = \{A_i, A_j, A_0\}$. Therefore, the line through A_i and A_j can be represented by the involution g_{A_i,A_j}. See Figure 3. Hence, we can similarly represent the lines of *type*–II by some certain involutions.

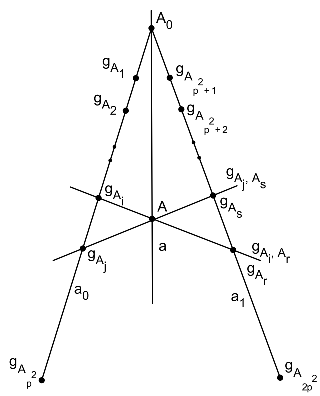

Figure 4: Determining lines of *type*–I by certain group elements of order p

Let A be the intersection point of the lines represented by the involutions g_{A_i,A_r} and g_{A_j,A_s}, where $i \neq j$, $1 \leq i,j \leq p^2$, and $r \neq s$, $p^2+1 \leq r,s \leq 2p^2$, respectively, and a the line of *type*–I passing through A_0 and A. Our computation shows that $Fix(g_{A_i,A_r}) \cap Fix(g_{A_j,A_s}) = \{A, A_0\}$ and $(a) = Fix(g_{A_i,A_r} g_{A_j,A_s})$, where $g_{A_i,A_r} g_{A_j,A_s} \in G_\pi$ is of order p. See Figure 4.

Proposition 2 *Let π be one of the planes α, β, or γ. Then π can be reconstructed from G_π.*

Proof: Let a be a line of *type*–II passing through A_i and A_j. Then $(a) = Fix(g_{A_i,A_j}) \setminus \{A_0\}$, where g_{A_i,A_j} is the involution representing a.

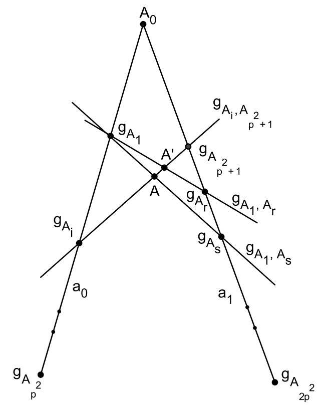

Figure 5: Determining lines of *type*–I by certain group elements of order p

Let A' and A be the intersection points of the line $g_{A_i,A_{p^2+1}}$ with lines g_{A_1,A_r} and g_{A_1,A_s}, where $r \neq s$, $p^2+2 \leq r,s \leq 2p^2$, and $2 \leq i \leq p^2$. It easily follows from the definition of a projective plane that A' and A are distinct points of *type*–II. See Figure 5. Let $i = 2$, then we have that $\{(a) \mid a \in (A_0)\} = \{Fix(g_{A_2,A_{p^2+1}} g_{A_1,A_r}) \mid p^2+2 \leq r \leq 2p^2\} \cup \{(a_0),(a_1)\}$.

The lines of π can be determined by the sets S_{ι,Syl_p} and S_{ι',Syl'_p} as described above. Hence, π can be reconstructed from G_π. $\Box$

6 Conclusion

"Is it always possible to construct projective planes from their collineation groups?". In [1] Brown shows how to construct both the Hughes plane of order q^2 and the Figueroa plane of order q^3, q is an odd prime power, from the linear group $GL(3,q)$. In our study we discuss a reconstruction method for a particular VW plane of order p^2, $p = 5, 7$, or 11. We show how to reconstruct the non–desarguesian VW planes α, β and γ, of orders 25, 49 and 121, respectively, from their collineation groups.

Acknowledgement

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Appendix

G_α

(i) $|G_\alpha| = 1,440,000 = 2^8\ 3^2\ 5^4$.

(ii) K has the following presentation:
$K = \langle x, y, a, b \mid x^{12}, a^5, b^5, aba^{-1}b^{-1}, y^2x^3, y^{-1}xy^3x^{10}, x^{-1}axb^2a^3, y^{-1}ayb^4a^2, x^{-1}bxa^3, y^{-1}byb^3a \rangle$.

(iii) Generators of the collineation group G_α:

x : (27, 47, 44, 28, 43, 32, 30, 35, 38, 29, 39, 50)(31, 34, 42, 36, 37, 33, 46, 48, 40, 41, 45, 49)
y : (27, 40, 29, 33, 30, 42, 28, 49)(31, 32, 41, 44, 46, 50, 36, 38)(34, 39, 45, 35, 48, 43, 37, 47)
a : (26, 41, 31, 46, 36)(27, 42, 32, 47, 37)(28, 43, 33, 48, 38)(29, 44, 34, 49, 39)(30, 45, 35, 50, 40)
b : (26, 49, 42, 40, 33)(27, 50, 43, 36, 34)(28, 46, 44, 37, 35)(29, 47, 45, 38, 31)(30, 48, 41, 39, 32)
m :
(6, 21)(7, 22)(8, 23)(9, 24)(10, 25)(11, 16)(12, 17)(13, 18)(14, 19)(15, 20)(31, 46)(32, 47)(33, 48)
(34, 49)(35, 50)(36, 41)(37, 42)(38, 43)(39, 44)(40, 45)

u : $\prod_{v=1}^{25} (v, v+25)$.

G_β

(i) $|G_\beta| = 22,127,616 = 2^{10}\ 3^2\ 7^4$.

(ii) K has the following presentation:
$K = \langle x, y, a, b \mid x^{24}, a^7, b^7, aba^{-1}b^{-1}, y^2x^{12}, y^{-1}xyx^{17}, x^{-1}axb^3a^2, y^{-1}ayb^6a^6, x^{-1}bxa^4, y^{-1}byba^2 \rangle$.

(iii) Generators of the collineation group G_β:

x :
(51, 77, 57, 94, 53, 68, 71, 84, 52, 97, 64, 89, 56, 79, 92, 62, 54, 88, 78, 72, 55, 59, 85, 67)
(58, 65, 60, 63, 74, 95, 73, 75, 66, 80, 70, 69, 98, 91, 96, 93, 82, 61, 83, 81, 90, 76, 86, 87)

y :
(51, 69, 56, 87)(52, 81, 55, 75)(53, 93, 54, 63)(57, 65, 92, 91)(58, 84, 98, 72)(59, 96, 97, 60)
(61, 71, 95, 78)(62, 90, 94, 66)(64, 80, 85, 76)(67, 74, 89, 82)(68, 86, 88, 70)(73, 77, 83, 79)

a :
(50, 84, 62, 89, 67, 94, 72)(51, 78, 63, 90, 68, 95, 73)(52, 79, 57, 91, 69, 96, 74)(53, 80, 58, 85, 70, 97, 75)
(54, 81, 59, 86, 64, 98, 76)(55, 82, 60, 87, 65, 92, 77)(56, 83, 61, 88, 66, 93, 71)

b :
(50, 64, 78, 92, 57, 71, 85)(51, 65, 79, 93, 58, 72, 86)(52, 66, 80, 94, 59, 73, 87)(53, 67, 81, 95, 60, 74, 88)
(54, 68, 82, 96, 61, 75, 89)(55, 69, 83, 97, 62, 76, 90)(56, 70, 84, 98, 63, 77, 91)

m :
(8, 43)(9, 44)(10, 45)(11, 46)(12, 47)(13, 48)(14, 49)(15, 36)(16, 37)(17, 38)(18, 39)(19, 40)(20, 41)(21, 42)
(22, 29)(23, 30)(24, 31)(25, 32)(26, 33)(27, 34)(28, 35)(57, 92)(58, 93)(59, 94)(60, 95)(61, 96)(62, 97)(63, 98)
(64, 85)(65, 86)(66, 87)(67, 88)(68, 89)(69, 90)(70, 91)(71, 78)(72, 79)(73, 80)(74, 81)(75, 82)(76, 83)(77, 84)

u : $\prod_{v=1}^{49} (v, v+49)$.

G_γ

(i) $|G_\gamma| = 843,321,600 = 2^8\ 3^2\ 5^2\ 11^4$.

(ii) K has the following presentation:
$K = \langle x, y, a, b \mid x^{60}, a^{11}, b^{11}, aba^{-1}b^{-1}, y^2x^{30}, y^{-1}xyx^{49}, x^{-1}axb^{-1}a^3, y^{-1}ayb^2a^{-1}, x^{-1}bxba^6, y^{-1}byba^{-1}\rangle$.

(iii) Generators of the collineation group G_γ:

$x:$
(2, 84, 80, 12, 70, 26, 3, 35, 38, 23, 18, 51, 5, 69, 75, 45, 24, 90, 9, 16, 17, 89, 47, 58, 6, 31, 33, 56, 93, 115, 11, 50, 54, 111, 64, 108, 10, 99, 96, 100, 116, 83, 8, 65, 59, 78, 110, 44, 4, 118, 117, 34, 87, 76, 7, 103, 101, 67, 41, 19)(13, 32, 105, 14, 104, 63, 25, 52, 88, 27, 86, 114, 49, 92, 43, 53, 39, 106, 97, 62, 74, 94, 77, 79, 61, 112, 15, 66, 21, 36, 121, 102, 29, 120, 30, 71, 109, 82, 46, 107, 48, 20, 85, 42, 91, 81, 95, 28, 37, 72, 60, 40, 57, 55, 73, 22, 119, 68, 113, 98)

$y:$
(2, 22, 11, 112)(3, 32, 10, 102)(4, 42, 9, 92)(5, 52, 8, 82)(6, 62, 7, 72)(12, 21, 111, 113)
(13, 31, 121, 103)(14, 41, 120, 93)(15, 51, 119, 83)(16, 61, 118, 73)(17, 71, 117, 63)(18, 81, 116, 53)
(19, 91, 115, 43)(20, 101, 114, 33)(23, 30, 100, 104)(24, 40, 110, 94)(25, 50, 109, 84)(26, 60, 108, 74)
(27, 70, 107, 64)(28, 80, 106, 54)(29, 90, 105, 44)(34, 39, 89, 95)(35, 49, 99, 85)(36, 59, 98, 75)
(37, 69, 97, 65)(38, 79, 96, 55)(45, 48, 78, 86)(46, 58, 88, 76)(47, 68, 87, 66)(56, 57, 67, 77)

$a:$
(1, 21, 30, 39, 48, 57, 77, 86, 95, 104, 113)(2, 22, 31, 40, 49, 58, 67, 87, 96, 105, 114)
(3, 12, 32, 41, 50, 59, 68, 88, 97, 106, 115)(4, 13, 33, 42, 51, 60, 69, 78, 98, 107, 116)
(5, 14, 23, 43, 52, 61, 70, 79, 99, 108, 117)(6, 15, 24, 44, 53, 62, 71, 80, 89, 109, 118)
(7, 16, 25, 34, 54, 63, 72, 81, 90, 110, 119)(8, 17, 26, 35, 55, 64, 73, 82, 91, 100, 120)
(9, 18, 27, 36, 45, 65, 74, 83, 92, 101, 121)(10, 19, 28, 37, 46, 66, 75, 84, 93, 102, 111)
(11, 20, 29, 38, 47, 56, 76, 85, 94, 103, 112)

$b:$
(1, 5, 9, 2, 6, 10, 3, 7, 11, 4, 8)(12, 16, 20, 13, 17, 21, 14, 18, 22, 15, 19)(23, 27, 31, 24, 28, 32, 25, 29, 33, 26, 30)(34, 38, 42, 35, 39, 43, 36, 40, 44, 37, 41)(45, 49, 53, 46, 50, 54, 47, 51, 55, 48, 52)(56, 60, 64, 57, 61, 65, 58, 62, 66, 59, 63)(67, 71, 75, 68, 72, 76, 69, 73, 77, 70, 74)(78, 82, 86, 79, 83, 87, 80, 84, 88, 81, 85)(89, 93, 97, 90, 94, 98, 91, 95, 99, 92, 96)(100, 104, 108, 101, 105, 109, 102, 106, 110, 103, 107)(111, 115, 119, 112, 116, 120, 113, 117, 121, 114, 118)

$m:$
(12, 111)(13, 112)(14, 113)(15, 114)(16, 115)(17, 116)(18, 117)(19, 118)(20, 119)(21, 120)(22, 121)
(23, 100)(24, 101)(25, 102)(26, 103)(27, 104)(28, 105)(29, 106)(30, 107)(31, 108)(32, 109)(33, 110)
(34, 89)(35, 90)(36, 91)(37, 92)(38, 93)(39, 94)(40, 95)(41, 96)(42, 97)(43, 98)(44, 99)(45, 78)(46, 79)
(47, 80)(48, 81)(49, 82)(50, 83)(51, 84)(52, 85)(53, 86)(54, 87)(55, 88)(56, 67)(57, 68)(58, 69)(59, 70)
(60, 71)(61, 72)(62, 73)(63, 74)(64, 75)(65, 76)(66, 77)

$u:\ \prod_{v=1}^{121}(v, v+121)$.